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Discrete sets with minimal moment of inertia ☆

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ABSTRACT

We analyze the moment of inertia $\mathrm{I}(S)$, relative to the center of gravity, of finite plane lattice sets S. We classify these sets according to their roundness: a set S is rounder than a set T if $\mathrm{I}(S) < \mathrm{I}(T)$. We introduce the notion of quasi-discs and show that roundest sets are strongly-convex quasi-discs in the discrete sense. We use weakly unimodal partitions and an inequality for the radius to make a table of roundest discrete sets up to size 40. Surprisingly, it turns out that the radius of the smallest disc containing a roundest discrete set S is not necessarily the radius of S as a quasi-disc.

1. Introduction

In this paper we consider plane sets up to translations. By a *discrete set* we mean a finite set of lattice points or a finite union of lattice closed unit squares (*pixels*) (Fig. 1 (a)). In particular, the word *polyomino* is defined as a 4-connected finite union of pixels in the plane. This means the neighboring squares are required to share an edge (Fig. 1(b)). These sets are well-known combinatorial objects in discrete geometry. The *dual* of a discrete set of pixels is the set of the centers of its pixels. The dual of a polyomino is usually called an *animal* (Fig. 1(c)). Using a $(\frac{1}{2}, \frac{1}{2})$ shift, we can always assume that the dual of a discrete set of pixels is a subset of the discrete plane $\mathbb{Z} \times \mathbb{Z}$.

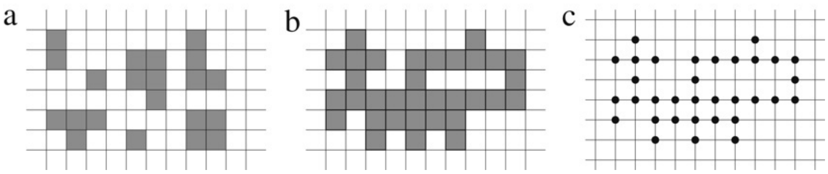

Fig. 1.

Moreover, a polyomino is called *v-convex* (resp. *h-convex*) if all its columns (resp. rows) are connected (see Fig. 2(a), (b)). We say that a polyomino is *hv-convex* (Fig. 2(c)) if all its columns and rows are connected and *strongly-convex* (Fig. 2(d)) if given any two points u and v in its corresponding animal, the lattice points in the segment $[u, v]$ are all in the animal. In the case of polyominoes, as Kim [1,2] showed, this notion coincides with the MP-convexity of Minsky and Papert [3] since animals are 4-connected. The goal of this paper is to study the *roundest* discrete sets S of N pixels (or N points), over the class of all discrete sets with the same number of pixels (or points), in the sense of having minimal moment of inertia $\mathrm{I}(S)$, relative to the center of gravity. This problem was raised in previous papers [4,5] in the context of the study of incremental

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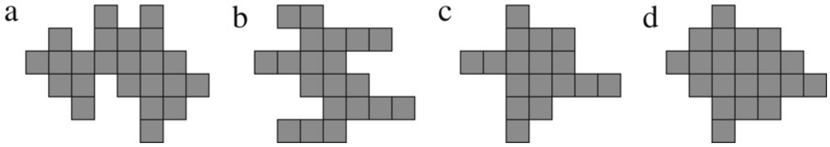


Fig. 2. A polyomino (a) v-convex (b) h-convex (c) hv-convex (d) strongly-convex.

algorithms based on the discrete Green theorem. The present notion of roundness is distinct from the one given in [6] where they consider minimizing the *site perimeter* of lattice sets, that is the number of points with *Manhattan* distance 1 from the sets. For a given N, Eq. (2) below ensures that minimizing $\mathrm{I}(S)$ is equivalent to minimizing $\mathrm{I}(A)$, where A is the dual of the discrete set S. In this paper, the Euclidean distance $\sqrt{(u_1-v_1)^2+(u_2-v_2)^2}$ between two points $u=(u_1,u_2)$ and $v=(v_1,v_2)$ in the plane, is denoted by $|u-v|$.

In Section 2, we recall some basic notions about the moment of inertia of discrete sets. Section 3 is devoted to properties of roundest discrete sets. More precisely, we introduce the notion of *discrete quasi-disc* and prove that roundest discrete sets are animals which are strongly-convex quasi-discs. Then a method is developed for computing the roundest discrete sets according to size ($N \le 40$) and some parameters associated to them. Finally, we briefly indicate in Section 4, how to extend our results to other kinds of lattices.

2. Continuous and discrete moments of inertia

We recall definitions of the basic geometric parameters:

Definition 1. Let S be a measurable subset of the real plane $\mathbb{R}\times\mathbb{R}$ such that

$$\int\int_S |z|^2 \mathrm{d}x\,\mathrm{d}y < \infty. \tag{1}$$

The *center of gravity* $g=g(S)$ and the *moment of inertia* $\mathrm{I}=\mathrm{I}(S)$ relative to the center of gravity, of the set S, are defined by the following equations:

$$g=g(S)=\frac{1}{\mathrm{Area}(S)}\int\int_S z\mathrm{d}x\,\mathrm{d}y$$

and

$$\mathrm{I}=\mathrm{I}(S)=\int\int_S |z-g|^2\mathrm{d}x\,\mathrm{d}y=\int\int_S |z|^2\mathrm{d}x\,\mathrm{d}y-\frac{1}{\mathrm{Area}(S)}\left|\int\int_S z\mathrm{d}x\,\mathrm{d}y\right|^2,$$

where

$$\mathrm{Area}(S)=\int\int_S \mathrm{d}x\,\mathrm{d}y.$$

Note that, in particular, if $S=P_1\cup\cdots\cup P_N$ is a union of N distinct pixels, the condition

$$\int\int_{P_1\cup P_2\cup\cdots\cup P_N} |z|^2\mathrm{d}x\ \mathrm{d}y<\infty$$

is obviously satisfied and $g(S)$ and $\mathrm{I}(S)$ are well-defined.

Note also that the moment of inertia of any single pixel P is $\mathrm{I}(P)=\frac{1}{6}$ and its center of gravity corresponds to its geometrical center.

Definition 2. Let $T=\{a_1,\ldots,a_N\}\subseteq\mathbb{Z}\times\mathbb{Z}$ be a set of N distinct points in the discrete plane where the point a_k has a non-negative mass m_k, for $k=1,\ldots,N$. The center of gravity $g=g(T)$ and the moment of inertia $\mathrm{I}=\mathrm{I}(T)$ relative to the center of gravity, of the set T are defined by

$$g=g(T)=\frac{1}{m_1+\cdots+m_N}\sum_{k=1}^{N} m_k a_k,$$

and

$$\begin{aligned}\mathrm{I}=\mathrm{I}(T)=\sum_{k=1}^{N} m_k|a_k-g|^2 &=\sum_{k=1}^{N} m_k|a_k|^2-\frac{1}{m_1+\cdots+m_N}\left|\sum_{k=1}^{N} m_k a_k\right|^2\\ &=\frac{1}{m_1+\cdots+m_N}\sum_{k<l} m_k m_l|a_k-a_l|^2.\end{aligned}$$

It is easily checked that the following relation holds for any discrete set:

Lemma 3. *Let $S = P_1 \cup P_2 \cup \cdots \cup P_N$ be a union of N distinct pixels $P_1, \ldots, P_N$ and the set A of their centers, $N = |A|$. Then,*

$$\mathrm{I}(S) = \mathrm{I}(A) + \frac{N}{6}. \tag{2}$$

A straightforward computation of the moment of inertia of a set of n equidistant points yields the following formula which will be used in Section 3.2 for the computation of discrete sets of minimal moment of inertia.

Lemma 4. *Let A be a set of n equidistant points, each of unit mass, on a line. Then,*

$$\mathrm{I}(A) = \frac{n^3 - n}{12}\, d^2 \tag{3}$$

where d is the distance between successive points.

3. Properties of roundest discrete sets

From now on, we assume that every measurable subset of $\mathbb{R} \times \mathbb{R}$, satisfies (1). The next lemma generalizes (2) and is a consequence of the classical parallel axis theorem [7] stating that, for any point p and any measurable set S, the moment of inertia of S relative to p, denoted by $\mathrm{I}_p(S)$ and defined by

$$\mathrm{I}_p(S) = \int\int_S |z-p|^2 \mathrm{d}x\,\mathrm{d}y \quad \left(\text{or } \sum_{k=1}^{N} m_k |a_k - p|^2\right),$$

satisfies

$$\mathrm{I}_p(S) = \mathrm{I}(S) + m|p-g|^2, \tag{4}$$

where $g = g(S)$ and m is the mass of S.

Lemma 5. *Let $S_1, S_2, \ldots, S_N$ be a sequence of disjoint measurable (or finite) subsets $\subseteq \mathbb{Z} \times \mathbb{Z}$. Then*

$$\begin{aligned}
\mathrm{I}(S_1 \cup \cdots \cup S_N) &= \sum_{k=1}^{N} \mathrm{I}(S_k) + \mathrm{I}(\{g_1, \ldots, g_N\}) \\
&= \sum_{k=1}^{N} \mathrm{I}(S_k) + \sum_{k=1}^{N} m_k |g_k|^2 - \frac{1}{m}\left|\sum_{k=1}^{N} m_k g_k\right|^2 \\
&= \sum_{k=1}^{N} \mathrm{I}(S_k) + \frac{1}{m}\sum_{k<l} m_k m_l |g_k - g_l|^2,
\end{aligned}$$

where g_k is the center of gravity of S_k with mass $m_k = \int\int_{S_k} \mathrm{d}x\,\mathrm{d}y$.

Proof. By (4), we can write, for any $p \in \mathbb{Z} \times \mathbb{Z}$,

$$\mathrm{I}_p(S_k) = \mathrm{I}(S_k) + m_k|p - g_k|^2, \quad k = 1, \ldots, N.$$

By additivity of I_p, we have

$$\begin{aligned}
\mathrm{I}_p(S_1 \cup \cdots \cup S_N) &= \mathrm{I}_p(S_1) + \cdots + \mathrm{I}_p(S_N) \\
&= \mathrm{I}(S_1) + \cdots + \mathrm{I}(S_N) + m_1|p-g_1|^2 + \cdots + m_N|p-g_N|^2.
\end{aligned}$$

Taking $p = g(S_1 \cup \cdots \cup S_N) = (m_1 g_1 + \cdots + m_N g_N)/m$, we get

$$\begin{aligned}
\mathrm{I}(S_1 \cup \cdots \cup S_N) &= \mathrm{I}_g(S_1 \cup \cdots \cup S_N) \\
&= \mathrm{I}(S_1) + \cdots + \mathrm{I}(S_N) + m_1|g-g_1|^2 + \cdots + m_N|g-g_N|^2 \\
&= \mathrm{I}(S_1) + \cdots + \mathrm{I}(S_N) + \mathrm{I}\left(\{g_1, \ldots, g_N\}\right). \quad \square
\end{aligned}$$

Note that formula (2) corresponds to the special case $S_1 = P_1, \ldots, S_N = P_N$ with $g_k = g(P_k)$. More precisely,

$$\mathrm{I}(S) = \mathrm{I}(P_1 \cup \cdots \cup P_N) = \sum_{k=1}^{N} \mathrm{I}(P_k) + \mathrm{I}(\{g_1, \ldots, g_N\}) = \frac{N}{6} + \mathrm{I}(A),$$

since $\mathrm{I}(P_k) = \frac{1}{6}$ and A is the corresponding set of centers of gravity of the pixels of S.

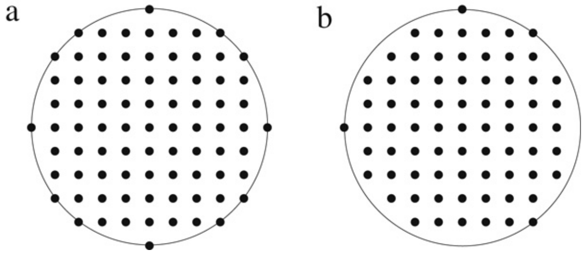


Fig. 3. (a) A (discrete) disc and (b) a (discrete) quasi-disc.

3.1. Roundest discrete sets are quasi-discs

From now on we call *roundest* a set having minimal moment of inertia among the discrete sets consisting of a fixed number N of points. In order to analyze properties of roundest sets we introduce the notion of discrete quasi-disc.

Definition 6. Let $c \in \mathbb{R} \times \mathbb{R}$, and $S \subseteq \mathbb{Z} \times \mathbb{Z}$ be a finite set of lattice points. Then S is called a:

(i) (discrete) *disc* centered at c of radius r if

$$S = \{z : \ |z - c| \leq r\} \cap (\mathbb{Z} \times \mathbb{Z}),$$

(ii) (discrete) *quasi-disc* centered at c of radius r if

$$\{z : \ |z - c| < r\} \cap (\mathbb{Z} \times \mathbb{Z}) \subseteq S \subseteq \{z : \ |z - c| \leq r\} \cap (\mathbb{Z} \times \mathbb{Z}),$$

where $r = \max_{s \in S} |s - c|$.

A disc and a quasi-disc of radius $r = 5$ are shown in Fig. 3 (a) and (b) respectively. Note that every lattice point on the circumference must belong to a disc while at least only one is necessary in the case of a quasi-disc. In both cases, every lattice point lying within the circumference must belong to the disc and quasi-disc.

Lemma 7. *Let A be a quasi-disc. Then:*

(i) *A is a strongly-convex set;*
(ii) *if A contains $N \neq 2$ points, then A is an animal.*

Proof. (i) Let A be a quasi-disc of radius r centered at c. Given any two distinct points $u, v \in A$, any lattice point $w \in [u, v]$, $w \neq u$, $w \neq v$, is necessarily in the topological interior of the disc $\{z : |z - c| \leq r\}$. Hence, by Definition 6, $w \in A$.

(ii) For $N = 1$ the result is obvious, and for $N = 2$ it is false since the quasi-disc $\{(0, 0), (1, 1)\}$ is not an animal. Let A be a quasi-disc of $N \geq 3$ points with radius $r > 0$ centered at $c \in \mathbb{R} \times \mathbb{R}$. Let u, v be two distinct lattice points in A. We must show that there is a path γ from u to v made only of vertical and horizontal unit steps. If u and v are both on a vertical or horizontal segment then by (i), we can choose γ as the linear path from u to v. Otherwise, u and v are not on a same vertical or horizontal segment, and we consider the lattice rectangle R with vertices u, v', v, u', given by

$$u = (u_1, u_2), \qquad v' = (v_1, u_2), \qquad v = (v_1, v_2), \qquad u' = (u_1, v_2).$$

Note that,

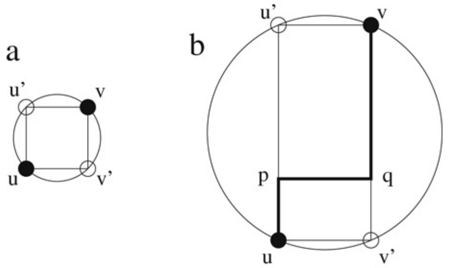
$$2r^2 \geq |u - c|^2 + |v - c|^2 = |u' - c|^2 + |v' - c|^2. \tag{5}$$

If at least one of the lattice points u' or v', say p, is in A, then since A is a quasi-disc, we can choose, for γ, the L-shaped path $([u, p] \cup [p, v]) \cap \mathbb{Z} \times \mathbb{Z}$ from u to v.

Taking (5) into account, the remaining possibility is that u' and v' are both on the circumference $|z - c| = r$. In other words

$$|u' - c|^2 = |v' - c|^2 = r^2.$$

This means that $|z - c| = r$ is the circumcircle of the rectangle R having vertices $u \in A$, $v' \notin A$, $v \in A$, $u' \notin A$. In this case, R cannot be a unit square (see Figure (a)) since $N \geq 3$. Hence, R is a $m \times n$-rectangle with $m > 1$ or $n > 1$ and there obviously exists a path γ from u to v made only of vertical and horizontal unit steps (see Figure (b)).



□

Theorem 8. *Let S be a roundest discrete set of N pixels. Then S is a polyomino whose associated animal A is a quasi-disc centered at $g = g(A)$ with radius $r = \max_{a \in A} |a - g|$.*

Proof. Let A be the dual of a roundest discrete set S. Since the result is obviously true for $N \leq 2$, we assume that $N \geq 3$. Take $a_0 \in A$ such that

$$r = |a_0 - g| = \max_{a \in A} |a - g|$$

and consider the closed disc

$$\Gamma_{a_0} = \{z \in \mathbb{R} \times \mathbb{R} : |z - g_{a_0}| \leq |a_0 - g_{a_0}|\}, \quad \text{where } g_{a_0} = g(A \setminus \{a_0\}).$$

Let us prove first that every lattice point in the interior of the disc Γ_{a_0} is in A. This is a consequence of Lemma 5 with $N = 2$:

$$\mathrm{I}(S_1 \cup S_2) = \mathrm{I}(S_1) + \mathrm{I}(S_2) + \frac{m_1 m_2}{m_1 + m_2} |g_1 - g_2|^2,$$

with $S_1 = \{a_0\},\ S_2 = A \setminus \{a_0\},\ g_1 = a_0,\ g_2 = g_{a_0},\ m_1 = 1,\ m_2 = N - 1$. Then,

$$\begin{aligned} \mathrm{I}(A) &= \mathrm{I}(\{a_0\}) + \mathrm{I}(A \setminus \{a_0\}) + \frac{N-1}{N} \left|a_0 - g_{a_0}\right|^2 \\ &= \mathrm{I}(A \setminus \{a_0\}) + \frac{N-1}{N} \left|a_0 - g_{a_0}\right|^2 \end{aligned}$$

since $\mathrm{I}(\{a_0\}) = 0$. Assume now that Γ_{a_0} contains in its interior, a lattice point $b \notin A$, that means $|b - g_{a_0}| < |a_0 - g_{a_0}|$. Replace a_0 by b and consider the set $B = ((A \setminus \{a_0\}) \cup \{b\})$. Then,

$$\mathrm{I}(B) = \mathrm{I}((A \setminus \{a_0\}) \cup \{b\}) = \mathrm{I}(A \setminus \{a_0\}) + \frac{N-1}{N} |b - g_{a_0}|^2 < \mathrm{I}(A),$$

which contradicts the minimality of the moment of inertia of A. Consider now the closed disc

$$C_{a_0} = \{z \in \mathbb{R} \times \mathbb{R} : |z - g| \leq r\}.$$

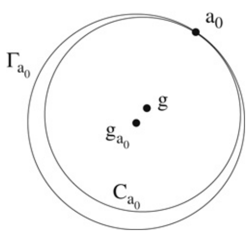
Then, by definition of C_{a_0}, we obviously have,

$$A \subseteq C_{a_0} \cap (\mathbb{Z} \times \mathbb{Z}). \tag{6}$$

Furthermore, it is easy to check that

$$g - a_0 = \frac{N-1}{N} (g_{a_0} - a_0).$$

This implies that g belongs to the segment $[g_{a_0}, a_0]$. Hence, $C_{a_0} \subseteq \Gamma_{a_0}$ as we can see in the following figure:



But we have seen that every lattice point in the interior of Γ_{a_0} is in A. In particular all those in the interior of C_{a_0} must be also in A:

$$(\mathrm{int}\ C_{a_0}) \cap (\mathbb{Z} \times \mathbb{Z}) \subseteq A.$$

Using (6), this implies that A is a quasi-disc of radius r, centered at g. Finally, by Lemma 7, A is a strongly-convex animal, since $N \geq 3$. □

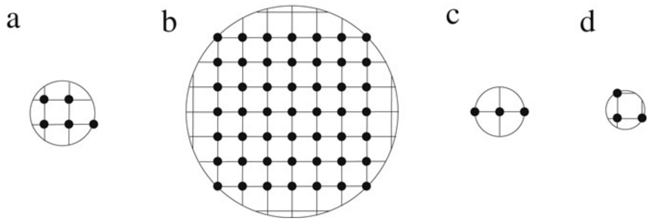


Fig. 4. Examples and counter examples for Theorem 8.

Note that the above proof shows that A is also a quasi-disc centered at g_{a_0} with radius $|a_0 - g_{a_0}| > r$.

Fig. 4 (a) illustrates Theorem 8 with $N = 5$. By contraposition, the 7×7 lattice set A in Fig. 4 (b) is not minimal since the disc C_{a_0} contains lattice points not in A. Note that the converse of Theorem 8 is false since, for $N = 3$, the quasi-disc of Fig. 4 (c) is not minimal (with $\mathrm{I} = 2$). The minimal one for $N = 3$ (with $\mathrm{I} = \frac{4}{3} < 2$) is shown in Fig. 4(d).

To pursue our study of roundest discrete sets we now give an upper bound for the radius r of the disc C_{a_0} as a function of the size N. This result reduces the number of candidates to be tested in the search of roundest animals in Section 3.2.

Lemma 9. *Let A be a roundest animal having N points. Then, the radius $r = |a_0 - g| = \max_{a \in A}|a - g|$ of the disc C_{a_0} centered at $g = g(A)$ satisfies*

$$r \leq \frac{1}{\sqrt{2}} + \sqrt{\frac{N}{\pi}}.$$

Proof. Let S be the polyomino associated to the animal A. We will show that the open disc $B^\circ(g, r - \frac{1}{\sqrt{2}})$, of radius $r - \frac{1}{\sqrt{2}}$, centered at g satisfies

$$B^\circ\left(g, r - \frac{1}{\sqrt{2}}\right) \subseteq S \tag{7}$$

and the result will follow since (7) implies that

$$\pi\left(r - \frac{1}{\sqrt{2}}\right)^2 \leq \text{ area } (S) = N.$$

To establish (7), consider an arbitrary point $z \in B^\circ(g, r - \frac{1}{\sqrt{2}})$. We must show that there exists $\nu \in A$ such that $z \in \mathrm{pix}_\nu$, where pix_ν is the pixel centered at ν. So, let $z = (x, y)$ be such that

$$|z - g| < r - \frac{1}{\sqrt{2}}.$$

Then there exist integers ν_1, ν_2 such that,

$$x = \nu_1 + f_1, \qquad y = \nu_2 + f_2$$

where $|f_1| \leq \frac{1}{2}$, $|f_2| \leq \frac{1}{2}$. Let $\nu = (\nu_1, \nu_2)$ and $f = (f_1, f_2)$. We have $z = \nu + f \in \mathrm{pix}_\nu$. There remains to show that $\nu \in A$. We have, by the triangular inequality

$$|\nu - g| - |f| \leq |\nu - g + f| = |z - g| < r - \frac{1}{\sqrt{2}}.$$

Hence,

$$|\nu - g| < r - \frac{1}{\sqrt{2}} + |f| \leq r - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = r$$

since $|f| = \sqrt{f_1^2 + f_2^2} \leq \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$. By Theorem 8, we conclude that $\nu \in A$ since every lattice point in the open disc $B^\circ(g, r)$ necessarily belongs to A. $\square$

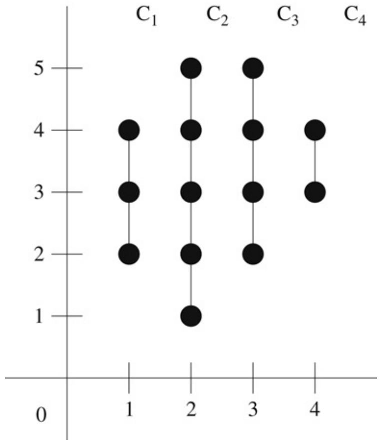


Fig. 5. Column C_k is the set of points in S above the point $(k, 0)$.

3.2. Computation of the roundest discrete sets according to size

In order to generate all the roundest animals of a given size N, we classify animals according to the sequence $(n_1, \ldots, n_s)$ of their vertical projections, where n_k is the number of points in the kth column for $k = 1, \ldots, s$ and $n_1 + \cdots + n_s = N$. If n_k is even (resp. odd), the kth column is said to be even (resp. odd). Let $\mathbb{A}_{n_1,\ldots,n_s}$ be the set of all animals having the sequence $(n_1, n_2, \ldots, n_s)$ of vertical projections with $N = n_1 + n_2 + \cdots + n_s$. We first characterize animals $A \in \mathbb{A}_{n_1,\ldots,n_s}$ such that

$$\mathrm{I}(A) = \min\{\mathrm{I}(B) : B \in \mathbb{A}_{n_1,\ldots,n_s}\}. \tag{8}$$

The roundest animals of size N are those that minimize $\mathrm{I}(A)$ under the condition $n_1 + \cdots + n_s = N$. Due to the convexity properties of roundest animals, we can reduce our analysis to sequences $(n_1, n_2, \ldots, n_s)$ satisfying

$$0 < n_1 \leq n_2 \leq \cdots \leq n_k < n_{k+1} = \cdots = n_{l-1} > n_l \geq \cdots \geq n_{s-1} \geq n_s > 0.$$

Such sequences are called *weakly unimodal partitions* of N (or *stack* or *planar partitions* of N), see Stanley [8]. We have the following result.

Lemma 10. *Let $(n_1, \ldots, n_s)$ be a weakly unimodal partition of N. Then,*

(i) *if $n_1, \ldots, n_s$ have the same parity, there is a unique animal $A \in \mathbb{A}_{n_1,\ldots,n_s}$, up to translation, with minimal moment of inertia over $\mathbb{A}_{n_1,\ldots,n_s}$. This animal is symmetric relative to an horizontal axis;*
(ii) *otherwise, there are exactly two animals $A, A' \in \mathbb{A}_{n_1,\ldots,n_s}$, up to translation, with minimal moment of inertia over $\mathbb{A}_{n_1,\ldots,n_s}$. These animals are symmetric to each other. The centers of gravity of the odd (resp. even) columns of each of these two animals, A and A', are all on a same horizontal axis X (resp. X'). The distance between X and X' is $1/2$.*

Moreover, the moment of inertia of A (and A') is given by the formula

$$\mathrm{I}(A) = \frac{1}{12}\sum_{k=1}^{s} n_k^3 - \frac{1}{12}N + \sum_{k=1}^{s} k^2 n_k - \frac{1}{N}\left(\sum_{k=1}^{s} k n_k\right)^2 + \frac{1}{4N}\left(\sum_{n_k \text{even}} n_k\right)\left(\sum_{n_k \text{odd}} n_k\right). \tag{9}$$

Proof. Let S be an animal with projections $(n_1, n_2, \ldots, n_s)$ having columns $C_1, C_2, \ldots, C_s$. More precisely, C_k is the column of points in S over the point $(k, 0)$, $k = 1, \ldots, s$ (see Fig. 5).

Let g_k be the center of gravity of C_k, $1 \leq k \leq s$. Note that there exists $\nu_k \in \mathbb{Z}$ such that, for $k = 1, \ldots, s$,

$$g_k = \begin{cases} (k, \nu_k) & \text{if } n_k \text{ is odd,} \\ (k, \nu_k + \frac{1}{2}) & \text{if } n_k \text{ is even.} \end{cases}$$

In other words,

$$g_k = \left(k, \nu_k + \frac{1}{2}\chi_{\text{even}}(n_k)\right) \quad k = 1, \ldots, s$$

where $\chi_{\text{even}}(n) = 1$, if n is even and 0 otherwise.

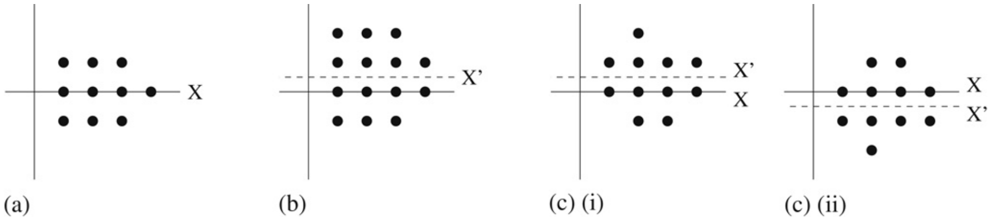


Fig. 6. (a) Every n_k is odd, (b) Every n_k is even, (c) Some n_k odd, some n_k even.

Then, by Lemma 5 and formula (3), we have

$$\begin{aligned}
\mathrm{I}(S) &= \mathrm{I}(C_1 \cup \cdots \cup C_s) \\
&= \mathrm{I}(C_1) + \cdots + \mathrm{I}(C_s) + \mathrm{I}(\{g_1, \cdots, g_s\}) \\
&= \sum_{k=1}^{s} \frac{n_k^3 - n_k}{12} + \frac{1}{N} \sum_{k<l} n_k n_l \left| \left(k-l,\, \nu_k - \nu_l + \frac{1}{2}(\chi_{\text{even}}(n_k) - \chi_{\text{even}}(n_l)) \right) \right|^2 \\
&= \frac{1}{12} \sum_{k=1}^{s} n_k^3 - \frac{1}{12} N + \frac{1}{N} \sum_{k<l} n_k n_l (k-l)^2 + \frac{1}{2N} \sum_{k,l} n_k n_l \left(\nu_k - \nu_l + \frac{1}{2}(\chi_{\text{even}}(n_k) - \chi_{\text{even}}(n_l)) \right)^2 \\
&= \frac{1}{12} \sum_{k=1}^{s} n_k^3 - \frac{1}{12} N + \sum_{k=1}^{s} k^2 n_k - \frac{1}{N} \left(\sum_{k=1}^{s} k n_k \right)^2 + \Omega,
\end{aligned}$$

where, after rearrangements,

$$\Omega = \frac{1}{2N} \left(\sum_{n_k \equiv n_l (\bmod 2)} n_k n_l (\nu_k - \nu_l)^2 + 2 \sum_{n_k \text{even}, n_l \text{odd}} n_k n_l \left(\nu_k - \nu_l + \frac{1}{2} \right)^2 \right).$$

This last expression attains its minimal value if and only if

$$\nu_k = \nu_l, \quad \text{whenever } n_k \equiv n_l \pmod 2$$

and

$$\left(\nu_k - \nu_l + \frac{1}{2} \right)^2 = \frac{1}{4}, \quad \text{whenever } n_k \text{ is even and } n_l \text{ is odd.}$$

In other words, the minimal value of Ω is attained if and only if, for some $p \in \mathbb{Z}$,

$$\nu_k = p \quad \text{for every } k, \text{ or } \begin{cases} \nu_k = p - 1 & \text{for even } n_k, \\ \nu_l = p & \text{for odd } n_l. \end{cases}$$

Since the moment of inertia is invariant under translation, we can assume that $p = 0$ and we conclude that the structure of a roundest animal A with projections $(n_1, \ldots, n_s)$ falls into one of the following three exclusive cases:

Case 1. *If every n_k is odd, then $g_k = (k, 0)$ for $k = 1, \ldots, s$* (Fig. 6 (a)).

Case 2. *If every n_k is even, then $g_k = (k, \frac{1}{2})$ for $k = 1, \ldots, s$* (Fig. 6 (b)).

Case 3. *Otherwise, two subcases can occur*:

3.(i) $g_k = (k, 0)$, *for n_k odd and* $g_k = \left(k, \frac{1}{2}\right)$, *for n_k even* (Fig. 6(c)i),

3.(ii) $g_k = (k, 0)$, *for n_k odd and* $g_k = \left(k, -\frac{1}{2}\right)$, *for n_k even* (Fig. 6(c)ii).

Fig. 6 illustrates the situation. It is easily checked that the minimal value of Ω is given by

$$\Omega = \frac{1}{4N} \left(\sum_{n_k \text{ even}} n_k \right) \left(\sum_{n_k \text{ odd}} n_k \right),$$

which establishes (9) and concludes the proof. □

Using the computer algebra software Maple [9], we now generate all roundest animals with size $N \leq 40$. Our strategy is the following: we first encode the weakly unimodal sequences by

$$(\lambda, b, h, \mu)$$

where λ, μ are integer partitions and $b, h \in \mathbb{N}^*$. The sequences $(n_1, n_2, \ldots, n_s)$ are given by

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k < \underbrace{h = \cdots = h}_{b} > \mu_l \geq \mu_{l-1} \geq \cdots \geq \mu_1, \tag{10}$$

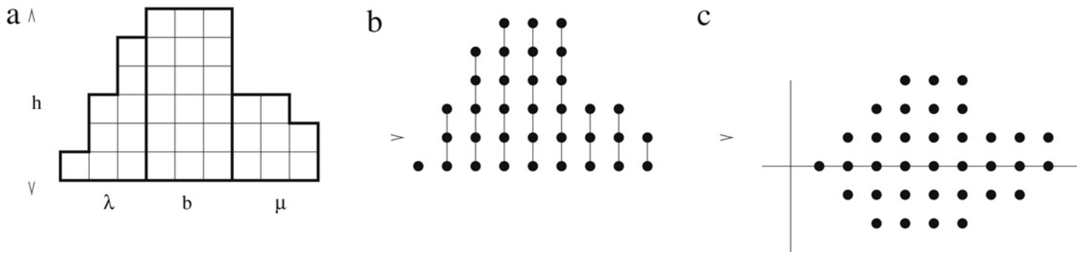


Fig. 7. (a) Encoding, (b) vertical projections, (c) animal to test, up to rotation.

Table 1
Roundest animals with size up to $N = 24$, up to symmetry

N	*Vertical projections*	$\mathrm{I}(A)$	$g(A)$	r	$c(A)$	$r_{\min}(A)$
1	[1]	0	$(1, 0)$	0	$(0, 0)$	0
2	[1, 1]	$\frac{1}{2}$	$(\frac{3}{2}, 0)$	$\frac{1}{2}$	$(\frac{3}{2}, 0)$	$\frac{1}{2}$
3	[1, 2]	$\frac{4}{3}$	$(\frac{5}{3}, \frac{1}{3})$	$\frac{1}{3}\sqrt{5}$	$(\frac{3}{2}, \frac{1}{2})$	$\frac{1}{2}\sqrt{2}$
4	[2, 2]	2	$(\frac{3}{2}, \frac{1}{2})$	$\frac{1}{2}\sqrt{2}$	$(\frac{3}{2}, \frac{1}{2})$	$\frac{1}{2}\sqrt{2}$
5a	[2, 2, 1]	4	$(\frac{9}{5}, \frac{2}{5})$	$\frac{2}{5}\sqrt{10}$	$(2, \frac{1}{2})$	$\frac{1}{2}\sqrt{5}$
5b	[1, 3, 1]	4	$(2, 0)$	1	$(2, 0)$	1
6	[2, 2, 2]	$\frac{33}{6}$	$(2, \frac{1}{2})$	$\frac{1}{2}\sqrt{5}$	$(2, \frac{1}{2})$	$\frac{1}{2}\sqrt{5}$
7	[2, 3, 2]	$\frac{52}{7}$	$(2, \frac{2}{7})$	$\frac{9}{7}$	$(2, \frac{1}{4})$	$\frac{5}{4}$
8	[3, 3, 2]	$\frac{78}{8}$	$(\frac{15}{8}, \frac{1}{8})$	$\frac{1}{8}\sqrt{130}$	$(2, 0)$	$\sqrt{2}$
9	[3, 3, 3]	$\frac{108}{9}$	$(2, 0)$	$\sqrt{2}$	$(2, 0)$	$\sqrt{2}$
10	[3, 3, 3, 1]	$\frac{156}{10}$	$(\frac{11}{5}, 0)$	$\frac{9}{5}$	$(\frac{7}{3}, 0)$	$\frac{5}{3}$
11a	[2, 4, 4, 1]	$\frac{212}{11}$	$(\frac{26}{11}, \frac{5}{11})$	$\frac{1}{11}\sqrt{349}$	$(\frac{5}{2}, \frac{1}{2})$	$\frac{1}{2}\sqrt{10}$
11b	[3, 4, 3, 1]	$\frac{212}{11}$	$(\frac{24}{11}, \frac{2}{11})$	$\frac{2}{11}\sqrt{101}$	$(\frac{9}{4}, \frac{1}{4})$	$\frac{5}{4}\sqrt{2}$
12	[2, 4, 4, 2]	$\frac{264}{12}$	$(\frac{5}{2}, \frac{1}{2})$	$\frac{1}{2}\sqrt{10}$	$(\frac{5}{2}, \frac{1}{2})$	$\frac{1}{2}\sqrt{10}$
13	[3, 4, 4, 2]	$\frac{340}{13}$	$(\frac{31}{13}, \frac{5}{13})$	$\frac{18}{13}\sqrt{2}$	$(\frac{23}{10}, \frac{3}{10})$	$\frac{13}{10}\sqrt{2}$
14	[3, 4, 4, 3]	$\frac{425}{14}$	$(\frac{5}{2}, \frac{2}{7})$	$\frac{3}{14}\sqrt{85}$	$(\frac{5}{2}, \frac{1}{6})$	$\frac{1}{6}\sqrt{130}$
15	[4, 4, 4, 3]	$\frac{528}{15}$	$(\frac{12}{5}, \frac{2}{5})$	$\frac{1}{5}\sqrt{113}$	$(\frac{5}{2}, \frac{1}{2})$	$\frac{3}{2}\sqrt{2}$
16a	[2, 4, 4, 4, 2]	$\frac{640}{16}$	$(3, \frac{1}{2})$	$\frac{1}{2}\sqrt{17}$	$(3, \frac{1}{2})$	$\frac{1}{2}\sqrt{17}$
16b	[4, 4, 4, 4]	$\frac{640}{16}$	$(\frac{5}{2}, \frac{1}{2})$	$\frac{3}{2}\sqrt{2}$	$(\frac{5}{2}, \frac{1}{2})$	$\frac{3}{2}\sqrt{2}$
17a	[4, 5, 5, 3]	$\frac{780}{17}$	$(\frac{41}{17}, \frac{2}{17})$	$\frac{40}{17}$	$(2, 0)$	$\sqrt{5}$
17b	[2, 4, 5, 4, 2]	$\frac{780}{17}$	$(3, \frac{6}{17})$	$\frac{40}{17}$	$(3, \frac{1}{6})$	$\frac{13}{6}$
18	[3, 4, 5, 4, 2]	$\frac{925}{18}$	$(\frac{26}{9}, \frac{5}{18})$	$\frac{1}{18}\sqrt{1685}$	$(3, 0)$	$\sqrt{5}$
19	[3, 5, 5, 4, 2]	$\frac{1084}{19}$	$(\frac{54}{19}, \frac{3}{19})$	$\frac{1}{19}\sqrt{1937}$	$(3, 0)$	$\sqrt{5}$
20	[3, 5, 5, 5, 2]	$\frac{1255}{20}$	$(\frac{29}{10}, \frac{1}{20})$	$\frac{1}{20}\sqrt{2165}$	$(3, 0)$	$\sqrt{5}$
21	[3, 5, 5, 5, 3]	$\frac{1428}{21}$	$(3, 0)$	$\sqrt{5}$	$(3, 0)$	$\sqrt{5}$
22	[3, 5, 5, 5, 4]	$\frac{1664}{22}$	$(\frac{34}{11}, \frac{1}{11})$	$\frac{21}{11}\sqrt{2}$	$(\frac{45}{14}, \frac{3}{14})$	$\frac{25}{14}\sqrt{2}$
23	[5, 5, 5, 5, 3]	$\frac{1916}{23}$	$(\frac{65}{23}, 0)$	$\frac{2}{23}\sqrt{970}$	$(\frac{21}{8}, 0)$	$\frac{5}{8}\sqrt{17}$
24	[1, 5, 5, 5, 5, 3]	$\frac{2183}{24}$	$(\frac{89}{24}, 0)$	$\frac{65}{24}$	$(\frac{18}{5}, 0)$	$\frac{13}{5}$

with $|\lambda| + bh + |\mu| = N$ (see Fig. 7). Then, using the package `combinat`, we generate all (λ, b, h, μ) such that the associated animal A minimizes the moment of inertia $\mathrm{I}(A)$ given by formula (9). Taking Lemma 3 into account, we restrict the generation of the 4-tuples (λ, b, h, μ) to those satisfying the further conditions

$$s \leq 2r + 1 \quad \text{and} \quad h \leq 2r + 1,$$

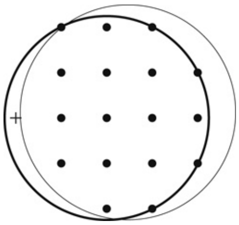
that is

$$\max(s, h) \leq \left\lfloor \sqrt{2} + 2\sqrt{\frac{N}{\pi}} + 1 \right\rfloor. \tag{11}$$

Fig. 8 gives, for each $N \leq 40$, a set of representatives, up to dihedral symmetry, of the roundest animals of size N. As an indication, for $N = 40$, we had to test 76 396 4-tuples (λ, b, h, μ) satisfying (10) and (11), instead of 4207 763 for those satisfying only (10).

Various parameters associated to these roundest animals ($N \leq 40$) are given in Tables 1 and 2. The first five columns give the size N, the vertical projections, the moment of inertia, the center of gravity and the radius of the disc C_{a_0} of the roundest animals, up to dihedral symmetry.

Remark. Let $C_{\min} = \{z \in \mathbb{R} \times \mathbb{R} : |z - c| \leq r_{\min}\}$, be the smallest closed disc containing a given roundest animal A of size N. The value of c and $r_{\min}$ are given in the last two columns of Table 1. One may think that A is a quasi-disc centered at c having radius $r_{\min}$. In other words, we can replace the disc C_{a_0} of Theorem 8 by $C_{\min}$. It turns out that this is false in general.



As an example, consider for $N = 17$ the roundest animal A in row 17a of Table 1. Its projections are $(4, 5, 5, 3)$ and the smallest closed disc $C_{\min}$ containing it has radius $r_{\min} = \sqrt{5}$ and center $c = (2, 0)$. In this case C_{a_0} has radius $r = \frac{40}{17}$ and center $g = \left(\frac{41}{17}, \frac{2}{17}\right)$.

Then A *is not* a quasi-disc of radius $r_{\min} = \sqrt{5}$. Indeed, the lattice point $+$ (cross) belongs to the open disc $C^{\circ}_{\min}$ but is not an element of A. It turns out that the only occurrence of such an animal, up to $N = 40$, is precisely A.

This suggests the following conjecture.

Conjecture 11. *There exists an infinite family of roundest animals A which is not a quasi-disc of radius $r_{\min}$, where $r_{\min}$ is the radius of the smallest closed disc containing A.*

4. Conclusion

While the recognition of digital straight lines is a well-understood problem [10], both from the Euclidean and combinatorial approaches, the recognition of circles is still a challenging problem in discrete geometry. The border of discrete quasi-discs introduced in Section 3.1, appears as a good candidate for a circle. This fact begs for a thorough study of their properties, perhaps useful for circle recognition. The above results can be extended to other families of lattices. For instance, in the context of regular triangular lattices, a discrete set S is a union of N distinct regular hexagonal cells and the set A of the centers of these hexagons satisfies the following formula, analogous to (2),

$$\mathrm{I}(S) = \mathrm{I}(A) + N \cdot \mathrm{I}(H)$$

where H is the fundamental hexagon of the lattice. The lattice set $\mathbb{Z}\times\mathbb{Z}$ must be replaced by the set $\mathbb{T} \subseteq \mathbb{R}\times\mathbb{R}$ of the centers of all hexagons. The associated notions of strong convexity, (discrete) disc and quasi-disc are easily defined using $\mathbb{T}$. Theorem 8 still holds. The radius estimate of Lemma 9 must be replaced by $r \leq \alpha + \beta\sqrt{N}$ with suitable constants α, β (according to the lattice). The computation of the roundest discrete sets can be established using an adaptation of the strategy described in Section 3.2. Moreover, extensions to higher dimensional lattices are also possible.

Table 2
Roundest animals with size up to $N = 40$, up to symmetry

N	*Vertical projections*	$\mathrm{I}(A)$	$g(A)$	r	$c(A)$	$r_{\min}(A)$
25	[3, 5, 5, 5, 5, 2]	$\frac{2474}{25}$	$\left(\frac{17}{5}, \frac{1}{25}\right)$	$\frac{1}{25}\sqrt{4801}$	$\left(\frac{7}{2}, 0\right)$	$\frac{1}{2}\sqrt{29}$
26	[3, 5, 5, 5, 5, 3]	$\frac{2769}{26}$	$\left(\frac{7}{2}, 0\right)$	$\frac{1}{2}\sqrt{29}$	$\left(\frac{7}{2}, 0\right)$	$\frac{1}{2}\sqrt{29}$
27	[1, 4, 6, 6, 6, 4]	$\frac{3116}{27}$	$\left(\frac{35}{9}, \frac{13}{27}\right)$	$\frac{13}{27}\sqrt{37}$	$\left(\frac{15}{4}, \frac{1}{2}\right)$	$\frac{5}{4}\sqrt{5}$
28	[4, 6, 6, 6, 4, 2]	$\frac{3464}{28}$	$\left(\frac{45}{14}, \frac{1}{2}\right)$	$\frac{1}{14}\sqrt{1570}$	$\left(\frac{13}{4}, \frac{1}{2}\right)$	$\frac{5}{4}\sqrt{5}$
29	[2, 5, 6, 6, 6, 4]	$\frac{3852}{29}$	$\left(\frac{108}{29}, \frac{12}{29}\right)$	$\frac{10}{29}\sqrt{74}$	$\left(\frac{7}{2}, \frac{1}{2}\right)$	$\frac{1}{2}\sqrt{34}$
30	[4, 6, 6, 6, 5, 3]	$\frac{4258}{30}$	$\left(\frac{101}{30}, \frac{11}{30}\right)$	$\frac{1}{30}\sqrt{7922}$	$\left(\frac{7}{2}, \frac{1}{2}\right)$	$\frac{1}{2}\sqrt{34}$
31	[3, 6, 6, 6, 6, 4]	$\frac{4688}{31}$	$\left(\frac{111}{31}, \frac{14}{31}\right)$	$\frac{1}{31}\sqrt{8642}$	$\left(\frac{7}{2}, \frac{1}{2}\right)$	$\frac{1}{2}\sqrt{34}$
32	[4, 6, 6, 6, 6, 4]	$\frac{5120}{32}$	$\left(\frac{7}{2}, \frac{1}{2}\right)$	$\frac{1}{2}\sqrt{34}$	$\left(\frac{7}{2}, \frac{1}{2}\right)$	$\frac{1}{2}\sqrt{34}$
33a	[4, 6, 6, 6, 6, 5]	$\frac{5680}{33}$	$\left(\frac{118}{33}, \frac{14}{33}\right)$	$\frac{80}{33}\sqrt{2}$	$\left(\frac{67}{18}, \frac{5}{18}\right)$	$\frac{41}{18}\sqrt{2}$
33b	[4, 6, 6, 6, 6, 4, 1]	$\frac{5680}{33}$	$\left(\frac{119}{33}, \frac{16}{33}\right)$	$\frac{80}{33}\sqrt{2}$	$\left(\frac{23}{6}, \frac{1}{2}\right)$	$\frac{1}{6}\sqrt{370}$
34	[4, 6, 7, 7, 6, 4]	$\frac{6241}{34}$	$\left(\frac{7}{2}, \frac{5}{17}\right)$	$\frac{1}{34}\sqrt{12\,833}$	$\left(\frac{7}{2}, \frac{1}{6}\right)$	$\frac{1}{6}\sqrt{370}$
35	[5, 6, 7, 7, 6, 4]	$\frac{6816}{35}$	$\left(\frac{24}{7}, \frac{8}{35}\right)$	$\frac{1}{35}\sqrt{13\,309}$	$\left(\frac{129}{38}, \frac{7}{38}\right)$	$\frac{1}{38}\sqrt{15\,170}$
36	[3, 5, 6, 7, 7, 5, 3]	$\frac{7406}{36}$	$\left(\frac{145}{36}, \frac{1}{12}\right)$	$\frac{1}{36}\sqrt{13\,546}$	$(4, 0)$	$\sqrt{10}$
37	[3, 5, 7, 7, 7, 5, 3]	$\frac{7992}{37}$	$(4, 0)$	$\sqrt{10}$	$(4, 0)$	$\sqrt{10}$
38	[4, 5, 7, 7, 7, 5, 3]	$\frac{8689}{38}$	$\left(\frac{149}{38}, \frac{1}{19}\right)$	$\frac{37}{38}\sqrt{13}$	$\left(\frac{23}{6}, \frac{1}{6}\right)$	$\frac{1}{6}\sqrt{410}$
39	[3, 5, 7, 7, 7, 6, 4]	$\frac{9388}{39}$	$\left(\frac{161}{39}, \frac{5}{39}\right)$	$\frac{1}{39}\sqrt{17\,873}$	$\left(\frac{25}{6}, \frac{1}{6}\right)$	$\frac{1}{6}\sqrt{410}$
40	[3, 6, 7, 7, 7, 6, 4]	$\frac{10127}{40}$	$\left(\frac{163}{40}, \frac{1}{5}\right)$	$\frac{1}{40}\sqrt{19\,433}$	$\left(\frac{235}{58}, \frac{15}{58}\right)$	$\frac{1}{58}\sqrt{39\,442}$

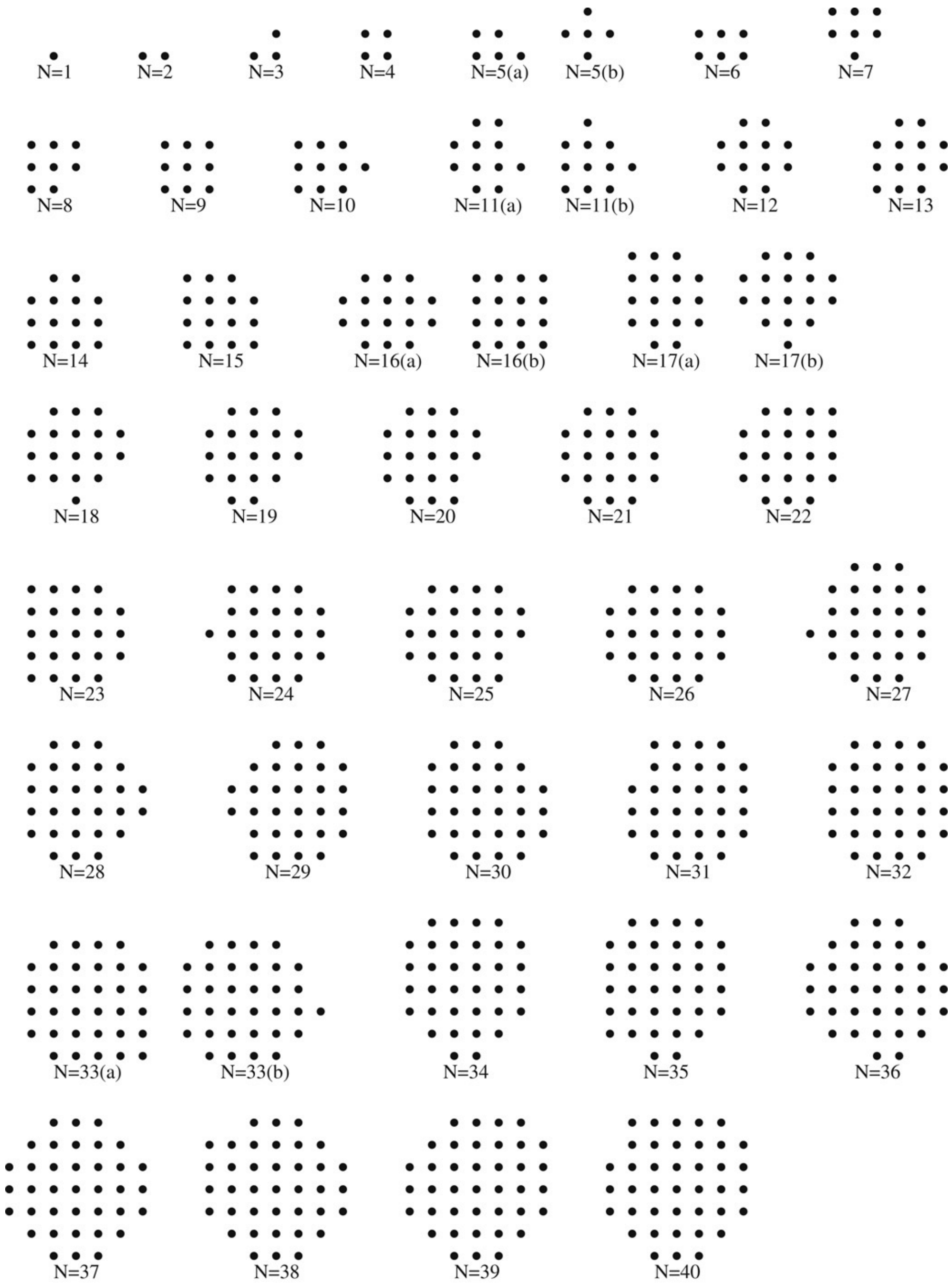


Fig. 8. The roundest animals of size $N \leq 40$ (up to dihedral symmetry).

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